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THEORY OF HEAT TRANSFER IN SMOOTH AND ROUGH PIPES

By G. D. Mattioli

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BASIC CONSIDERATIONS OF FRICTION TURBULENCE

Heat transfer theory in the turbulent region follows from the theory of turbulence described in a book by the author (reference 1). Prandtl (reference 2) postulates momentum transfer, Taylor postulates vorticity transfer, while Mattioli postulates both.

Prandtl's attack was based on the foundations of kinetic theory; the momentum transfer by the molecules explained the viscosity. But a large difference exists between molecular and turbulent motions. The molecules are small and their linear momentum is sufficient to explain their behavior, while the elementary units which partake of turbulent motion are large, and both their linear and angular momentum must be considered.

In the turbulent regime, the elementary units (which are not separated) will be considered as separate entities from a dynamical point of view. The momentum (per unit volume) ρv (ρ = mass density, v = velocity vector) must be augmented by the unit angular momentum $\rho l^2 \omega$, where $\omega = \text{rot } v$, which describes the vorticity of the mean motion, and l is a length which has been introduced on dimensional grounds.

Across each surface in the turbulent regime both linear momentum and angular momentum are transferred, and each exchange is dynamically equal to a fraction of the inner stresses.

Let x coincide with the tube axis; u is the velocity in the positive x-direction, r the radius from the tube axis, ϵ the eddy diffusivity, ρ the mass density; then $\rho \epsilon \frac{du}{dr}$ is the momentum transfer per unit area and time in

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*"Theorie der Wärmeübertragung in glatten und rauhen Rohren." Forschung auf dem Gebiete des Ingenieurwesens, Bd. 11, no. 4, July-August 1940, pp. 149-158.

the radial direction. This transfer is equal to the tangential shear*

$$\tau_{rx} = \rho\ \epsilon\ \frac{du}{dr}\ (kg/m^2) \tag{1}$$

Since the vorticity transfer of each element of mass does not vary, the corresponding stress tensor is symmetrical; that is, $\tau_{rx} = \tau_{xr}$. τ_{rx} is the x component of the stress, which acts on the element in a direction perpendicular to the r-axis. If $\tau_{rr} = \tau_{xx} = 0$, then the tensor is completely defined. The equation

$$\omega = -\frac{du}{dr} \tag{2}$$

is the vorticity of the mean motion. In view of the definition of ϵ, the difference between the vorticity transfer per unit time in the positive and negative directions of r is $2\pi\ r\ \rho\ \epsilon\ d(l^2\ \omega)/dr$, and

$$\frac{d}{dr}\left[2\pi\ r\ \rho\ \epsilon\ \frac{d(l^2\ \omega)}{dr}\right] dr$$

is the rate of increase of the vorticity between the radius r and $r + dr$. On a unit volume basis

$$M = \frac{1}{r}\ \frac{d}{dr}\left[r\ \rho\ \epsilon\ \frac{d(l^2\ \omega)}{dr}\right] \tag{3}$$

is the increase of vorticity per unit volume in a unit time. In the steady state the density of vorticity must be conserved. On the surface of each volume element ΔV stresses must exist which cause a moment about the center of the element of $-M\ \Delta\ V$ which destroys the point-to-point variation of the vorticity. About the axis of rotation of the element which coincides with the direction of the vortices, conditions are symmetrical; therefore, it is natural to postulate that σ_{rr}, σ_{rx}, etc., the components of the stress tensor are:

$$\sigma_{xx} = \sigma_{rr} = 0; \quad \sigma_{rx} = -\sigma_{xr} = M/2$$

This tensor is antisymmetrical. The viscosity generates inner forces, the corresponding tensor being:

*τ_{rx} denotes the x-component of the internal pressure acting on the surface element at right angles to the r-axis.

$$\upsilon_{xx} = \upsilon_{rr} = 0; \quad \upsilon_{rx} = \upsilon_{xr} = \rho\ \upsilon\ \frac{du}{dy}$$

where υ is the kinematic viscosity. The components of the total stress tensor then are:

$$t_{xx} = \tau_{xx} + \sigma_{xx} + \upsilon_{xx} = 0; \ t_{rr} = 0; \ t_{xr} = \rho(\epsilon + \upsilon)\ \frac{du}{dr}$$

$$- \frac{M}{2}; \ t_{rx} = \rho(\epsilon + \upsilon)\ \frac{du}{dr} + \frac{M}{2} \tag{4}$$

If ρ = constant and $\frac{dp}{dx}$ = constant, then

$$\frac{1}{r}\ \frac{d(r\ t_{rx})}{dr} = \frac{dp}{dx} = \text{constant} \tag{5}$$

for equilibrium on the element r to $r + dr$, of length dx. From these equations follow the equations of motion.

Employing equation (4):

$$\frac{1}{r}\ \frac{d}{dr}\left[\rho(\epsilon + \upsilon)\ r\ \frac{du}{dr} + \frac{rM}{2}\right] = \frac{dp}{dx} \tag{6}$$

and integrating, letting $\frac{1}{\rho}\ \frac{dp}{dx} = -\ \alpha$;

$$-\ \rho(\epsilon + \upsilon)\ \frac{du}{dr} - \frac{M}{2} = \rho\ \frac{\alpha r}{2} \tag{7}$$

For $r = 0$, the unit shear (left side of equation) is equal to zero, while on the wall $(r = R)$, the unit shear (τ_0) becomes $\rho\ \alpha\ \frac{R}{2}$.

The variable $v_* = \sqrt{\alpha\ \frac{R}{2}} = \sqrt{\frac{\tau_0}{\rho}}$ is called the friction velocity and can be evaluated at the wall.

A second equation results from the postulate which fixes M. Since the details of turbulent mixing are not yet known, a formal hypothesis with respect to M must suffice: A vortex can be generated only at the wall, for fluid elements striking the wall at other than zero tangential velocity and sticking there must experience a deformation which will cause an eddy to be generated. This vorticity generated at the walls must be destroyed in the interior of the fluid by the moment $-M$. But what is the law If $\rho\, l^2\, \omega$ is constant in the r-direction, then M would necessarily be zero. One may understandably let M be proportional to $d(\rho\, l^2\, \omega)/dr$. Walls are not present as vorticity generators in the case of free turbulence, and for this case $M = 0$.

Then for our case set

$$M = \psi \frac{d(\rho\, l^2\, \omega)}{dr} \tag{8}$$

and, together with equation (3), yields

$$\frac{1}{r}\frac{d}{dr}\left[r\, \rho\epsilon\, \frac{d(l^2\, \omega)}{dr}\right] = \psi\, \frac{d(\rho\, l^2\, \omega)}{dr} \tag{9}$$

Further considerations based on Prandtl's similarity theorem (in the neighborhood of the wall u/v_* = universal function of $(R - r)v_*/\nu$) allow ψ and l to be treated as constants, where

$$\psi = \chi\, v_*; \quad l^2 = 2\beta\, \frac{\nu^2}{\chi\, v_*^2} \tag{10}$$

and χ and β are universal constants of friction-generated turbulence. χ is the Kármán constant = 0.406 based on measurements of Nikuradse. (See reference 3.) β is harder to establish since it depends chiefly on the velocity distribution near the wall. It would appear that $2.46 < \beta < 3.5$.

THE HYDRODYNAMIC PROBLEM

Substitute (8) into (7) and utilize (10) yielding

$$(\epsilon + \nu)\, u' - \beta\, \frac{\nu^2}{v_*}\, u'' = -\frac{\alpha r}{2} \tag{11}$$

where $u' = \dfrac{du}{dr}$, $u'' = \dfrac{d^2u}{dr^2}$.

Integration of equation (9) yields

$$\epsilon\ r\ u'' = \chi\ v_*(r\ u' - u + K) \qquad (12)$$

where K is the integration constant which must be fixed by experiment. It is the only constant still open to question on theoretical grounds.

The boundary conditions which must be satisfied upon the integration of (11) and (12) follow from the Prandtl similarity theorem, and it seems natural to assume that the total unit shear at the wall depends on the viscosity, so that one may set:

$$\nu\ u' = \frac{\alpha R}{2} = v_*^2 \quad \text{for} \quad r = R$$

The velocity u at $r = R$ must also be introduced. The exchange (austausch) phenomenon at the wall is damped in such a thin layer that one may say that the inner interface of the fluid at which turbulent motion is just apparent is at $r = R$. From similarity considerations

$$u = \Delta\ v_* \quad \text{for} \quad r = R$$

where Δ is a universal dimensionless quantity which depends on β. From experiment

$$\Delta = 8.06 \quad \text{for} \quad \beta = 2.46$$

and

$$\Delta = 7.7 \quad \text{for} \quad \beta = 3.5$$

Subject to these boundary conditions the integrals of equations (11) and (12) yield the velocity and eddy diffusivity distributions.

For large values of $Re = \frac{2R\bar{u}}{\nu}$ ($\bar{u}$ = the mean velocity) equations (5) and (6) may be integrated by an approximate procedure, the result being correct for $Re \longrightarrow \infty$.

In the laminar sublayer where the velocity changes appreciably with distance, while the unit shear remains sensibly constant, r may be replaced by R. Let $R - r = y$, the distance from the wall, then

$$\varphi = \frac{u}{v_*}; \quad d\eta = \frac{v_*}{\nu}\, dy, \quad \eta = v_* \int_0^y \frac{dy}{\nu}^{1)} \tag{13}$$

If ν = constant, then $\eta = v_* y/\nu$. Since $\mu = F(r)$

$$\frac{dF}{dr} = -\frac{v_*}{\nu}\frac{dF}{d\eta}; \quad \frac{d^2F}{dr^2} = \frac{v_*}{\nu}\frac{d}{d\eta}\left(\frac{v_*}{\nu}\frac{dF}{d\eta}\right) = \frac{v_*^2}{\nu^2}\left(\frac{d^2F}{d\eta^2} - \frac{1}{\nu}\frac{d\nu}{d\eta}\frac{dF}{d\eta}\right)$$

and if the dots represent derivations with respect to η, equations (11) and (12) become in terms of the generalized variables $\varphi(= u^+)$, $\eta\ (= y^+)$

$$\left(\frac{\epsilon}{\nu} + 1\right)\dot{\varphi} + \beta\left(\ddot{\varphi} - \dot{\varphi}\,\frac{\dot{\nu}}{\nu}\right) = 1; \quad \frac{\epsilon}{\nu}\left(\ddot{\varphi} - \dot{\varphi}\,\frac{\dot{\nu}}{\nu}\right) = -\chi\left[\dot{\varphi} - \frac{\nu}{Rv_*^2}\,(K-u)\right]$$

For large magnitudes of Re, $(K - u)\,\frac{\nu}{Rv_*^2}$ is small compared with $\dot{\varphi}$ and for most fluids the influence of $\dot{\nu}$ is small, thus terms involving these two quantities will be neglected. Then the above equations become

$$\left(\frac{\epsilon}{\nu} + 1\right)\dot{\varphi} + \beta\,\ddot{\varphi} = 1 \quad \text{and} \quad \frac{\epsilon}{\nu}\,\ddot{\varphi} = -\chi\,\dot{\varphi} \tag{14 and 15}$$

which is the same form as ν = constant. That is, the influence of a variable ν is limited to the equation (13).

Integrals of (14) and (15) are fixed by the conditions at the wall, which in turn represent the momentum transfer mechanism at the wall. The transfer is dissipated in the "thin wall layer," which is to be thought of separately from the main stream as the "turbulent mixing mechanism" of the main stream goes over into the pattern at the wall in this "thin layer." The thickness of this layer is so small that it may be neglected and the wall process may be thought of as occurring in an interface. The point $y = 0$ ($\eta = 0$) is then the location of this interface (i.e., the inner surface of the boundary layer, where the word "inner" means the side toward the center) and the boundary conditions will be so applied to equations (14) and (15).

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[1]In the case of an isothermal flow the nondimensional distance from the wall is $\eta = v_* y/\nu$. For variable temperature ν varies considerably near the wall and this must be taken into account by a suitable expression.

The mechanism of the process in the boundary layer does not need to be known in detail, but it suffices to fix the velocity and the velocity gradient at the inner surface of the boundary layer. The Prandtl similarity theorem leads to the statement that

$$\varphi = \Delta \quad \text{and} \quad \dot{\varphi} = 1 \quad \text{for} \quad \eta = 0$$

For $\Delta = 8.06$ ($\beta = 2.46$) or $\Delta = 7.7$ (for $\beta = 3.5$) it follows from (14) and (15) that

$$\left(\frac{\epsilon}{\nu}\right)_{\eta=y=0} = \sqrt{\beta \chi} \tag{16}$$

and if ν is invariable

$$\left(\frac{d\epsilon}{dy}\right)_{y=0} = \frac{\sqrt{\beta\chi}}{2\chi} v_* = A\ v_* \tag{17}$$

that is, on the inner surface of the boundary layer the eddy diffusivity is not zero, which follows from the concept of the separate (from the main stream) boundary layer.

Equations (14) and (15) are integrable for $\beta \neq 0$ and yield a simple integral which is correct for $Re \longrightarrow \infty$, if the velocity distribution at the interface between the laminar sublayer and the turbulent stream is utilized. Substituting $\beta = 0$ and $\Delta = \Delta_0 = 9.46$, these integrals become in parametric form (see reference 1, p. 105):

$$\left.\begin{aligned} \varphi &= \frac{u}{v_*} = \frac{1}{\chi}\left(\tau - \ln \tau - 1\right) + \Delta_0 \\ \eta &= v_* \int_0^y \frac{dy}{\nu} = \frac{1}{\chi}\left(\frac{1}{\tau} + \ln \tau - 1\right); \quad \frac{\epsilon}{\nu} = \frac{1}{\tau} - 1 \end{aligned}\right\} \tag{18}$$

where $\tau = \dot{\varphi} = \dfrac{\nu}{v_*}\dfrac{d\varphi}{dy}$ and is less than 1 except at the wall where $\tau = 1$. Also $\dfrac{\epsilon}{\nu} = \dfrac{1}{\tau} - 1$ from (14) with $\beta = 0$.

Small values of τ describe the region in which the laminar sublayer transforms into the turbulent core. In this transition layer equation (18) may be simplified to

$$\frac{u}{v_*} = - \frac{1}{\chi} \ln \tau + \Delta_0 - \frac{1}{\chi}; \quad \eta - v_* \int_0^y \frac{dy}{\nu} = \frac{1}{\chi \tau} \tag{19}$$

which are asymptotic for $Re \longrightarrow \infty$, $\frac{\epsilon}{\nu} \longrightarrow \infty$ in the core for this case. In this region one may neglect ν compared to ϵ, and equations (11) and (12) become

$$\epsilon u' = - v_*^2 \frac{r}{R}; \quad \epsilon u'' = \chi v_* \left(u' - \frac{K - u}{r} \right) \tag{20}$$

the integrals of which agree with (19) for y 0. The hydrodynamic part has now been solved. The equations derived agree well with the experimental measurements of Nikuradse (reference 3).

THE THERMAL PROBLEM

Let

a thermal diffusivity

c_p unit heat capacity at constant pressure

$\gamma = g\rho$ weight density

ϑ temperature

The enthalpy per unit volume is $\gamma c_p \vartheta$ (k cal/m^3) and analogously with equation (3)

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \gamma c_p (\epsilon + a) \frac{\partial \vartheta}{\partial r} \right]$$

is the increase in heat content per unit time and volume in annular ring of width dr. If no sources exist

$$\gamma c_p \frac{\partial \vartheta}{\partial x} u = \frac{1}{r} \frac{\partial}{\partial r} \left[r \gamma c_p (\epsilon + a) \frac{\partial \vartheta}{\partial r} \right] \tag{21}$$

where $u(r)$ and $\epsilon(r)$ must be obtained from hydrodynamic considerations.

Equation (21) may be integrated in a manner similar to that employed for the hydrodynamic considerations. For instance, the rate of heat flow normal to the wall in the sublayer next to the wall is postulated as constant - which is similar to the statement that the unit shear is constant through the sublayer. Then

$$(\epsilon + a)\frac{\partial\vartheta}{\partial r} = \frac{-q}{\gamma\ c_p}$$

or, changing variables from r to η (equation (13))

$$\left(\frac{\epsilon}{\nu} + \frac{a}{\nu}\right)\frac{\partial\vartheta}{\partial\eta} = \frac{q}{\gamma\ c_p\ v_*} \tag{22}$$

where q is the heat rate per unit area normal to the wall measured positively in the direction of wall to fluid.

Substituting the new variable, τ from equation (18) yields

$$\left(\frac{1}{\tau} - 1 + \frac{a}{\nu}\right)\frac{\partial\vartheta}{\partial\tau} = -\frac{q}{\chi\ \gamma\ c_p\ v_*}\frac{1}{\tau}\left(\frac{1}{\tau} - 1\right) \tag{23}$$

For gases $\frac{a}{\nu}$ = constant, so that this equation is directly integrable. For liquids, even though a varies but little with temperature, ν cannot be considered invariable. It is seen that the influence of $\frac{a}{\nu}$ (< 0.5) in equation (23) is important only for large values of τ, so that to effect the formal integration, $\frac{a}{\nu}$ will be set equal to $\left(\frac{a}{\nu}\right)_0$ which is representative of the liquid temperature ϑ_0 at the wall. The integral of equation (23) becomes equal to ϑ_0, for $\tau = 1$. In general,

$$\frac{\gamma\ c_p\ v_*}{q}(\vartheta - \vartheta_0) = -\frac{1}{\chi}\left(\ln\ \tau + \frac{1}{P_{r_0}-1}\ln\ (P_{r_0}-(P_{r_0}-1)\tau)\right) \tag{24}$$

where $$Pr = \frac{\mu\ c_p\ g\ 3600}{k} = \frac{\nu}{a}$$

In the turbulent nuclear flow a, as ν, may be neglected with respect to ϵ. Then equation (21) becomes

$$\gamma\ c_p \frac{\partial \vartheta}{\partial x} r\ u = \frac{\partial}{\partial r}\left[\gamma\ c_p\ \epsilon\ r\ \frac{\partial \vartheta}{\partial r}\right] \tag{25}$$

where $u(r)$ and $\epsilon(r)$ are fixed by equation (20).

An approximate solution is now sought. The velocity distribution falls off so rapidly in the laminar sublayer that u may be held constant in the turbulent core and equal to the mean velocity $\overline{u}$. Then may be set

$$\frac{\partial \vartheta}{\partial x} = -\ n(x)$$

which is satisfied after the quieting length. Then from equation (25)

$$\gamma\ c_p\ \epsilon\ \frac{\partial \vartheta}{\partial r} = -\frac{1}{2}\ \gamma\ c_p\ \overline{u}\ R\ n(x)\ \frac{r}{R} \tag{26}$$

The left side describes the heat flow through the wall[2] $(-q)$ as r R; therefore,

$$q = \frac{1}{2}\ \gamma\ c_p\ R\ \overline{u}\ n(x)$$

and from equation (26)

$$\epsilon\ \frac{\partial \vartheta}{\partial r} = -\ \frac{q}{\gamma\ c_p}\ \frac{r}{R} \tag{27}$$

which is identical with equation (20). In the core

$$\frac{\gamma\ c_p\ v_*}{q}\ (\vartheta - \vartheta_1) = \frac{u - u_1}{v_*} \tag{28}$$

where the subscript 1 refers to any point; that is, the temperature and velocity profiles are similar in this region. The temperature ϑ_1 is fixed by the requirement that integral equation (28) in the buffer layer between the laminar sublayer and the turbulent core must become equal to equation (24).

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[2] It should be noted that the approximations are also valid if a is not neglected by comparison with ϵ, and that for $Re \longrightarrow \infty$ the thickness of the semilaminar boundary layer becomes vanishingly small. The above transition to the limit toward the inner side of the semilaminar boundary layer is thus identified with $r \longrightarrow R$.

TEMPERATURE DISTRIBUTION AT THE WALL

Since the numerical value of τ is small in the buffer layer, equation (24) becomes similar to equation (18) when applied here:

$$\frac{\gamma\ c_p\ v_*}{q}\ (\vartheta - \vartheta_o) = -\frac{1}{\chi}\left(\ln\ \tau + \frac{1}{P_{r_o} - 1}\ \ln\ P_{r_o}\right)$$

which, when compared with (19), may also be written as

$$\frac{\gamma\ c_p\ v_*}{q}\ (\vartheta - \vartheta_o) = \frac{u}{v_*} - \Delta_o + \frac{1}{\chi}\left(1 - \frac{1}{P_{r_o} - 1}\ \ln\ P_{r_o}\right) \tag{29}$$

Equation (28) presents a similar equation for the turbulent core; therefore, equation (29) yields the temperature distribution in the whole tube with the exception of a thin boundary layer whose thickness becomes zero for $Re \longrightarrow \infty$. It contains the unknown temperature (ϑ_o) of the fluid at the wall, which must be determined by the aid of thermal conditions at the wall. Referring to the velocity, the velocity increase is given as $\Delta\ v_*$, which occurs in the boundary layer where, for the approximation that $\beta = 0$, $\Delta = \Delta_o = 9.46$. Also, for the temperature a similar change must occur in the boundary layer. Dimensional considerations lead to the relation (reference 1, p. 306):

$$\theta = \vartheta_o - \vartheta_w = \Delta_\vartheta\ \frac{q}{\gamma\ c_p\ v_*}\ f(Pr_*) \tag{30}$$

Here ϑ_w is the wall temperature, Δ_ϑ is a constant, and $f(P_{r*})$ is a function of Pr which could be evaluated if the details of the heat transfer at the walls were known. As it is, it is clear that the function must be computed for an appropriate mean Pr_* for the boundary layer.

Even though the present theory will not allow the determination of $f(P_{r*})$ uniquely, its range may be established by means of a discussion of two limiting conditions, between which the true function must lie.

Say $\epsilon = 0$ at the inner surface of the boundary layer; then there is no mixing of the fluid in the boundary layer with that of the outer region (the core) and the heat transfer must be by pure conduction. Then

$$\frac{d\vartheta}{dy} = \frac{q}{\gamma\ c_p\ a}; \quad \theta = \frac{q}{\gamma\ c_p\ a}\ \delta$$

where δ is the thickness of the boundary layer. In the region near the wall ν/v_* is the significant length; therefore a likely relation is

$$\delta = \Delta_\vartheta\ \frac{\nu}{v_*}$$

from which there follows:

$$\theta = \Delta_\vartheta\ \frac{q}{\gamma\ c_p\ v_*}\ P_r \tag{31}$$

By comparison with (30), $f(P_r) = P_r$.

In accordance with equation (16) $|\epsilon|_{y=0}$ is not zero; that is, there is turbulent transfer from the inner side of the boundary layer which results in the introduction of new fluid at this interface. This fluid stream perpendicular to the interface is proportional (reference 1, p. 20) to $d\epsilon/dy$ which, at $y = 0$, is

$$\left|\frac{d\epsilon}{dy}\right|_{y=0} = A\ v_* \tag{32}$$

so that the quotient of the thickness $\Delta\ \frac{\nu}{v_*}$ of the boundary layer and the magnitude of equation (32) gives the mean time which a fluid element spends in the boundary layer:

$$T = \Delta\ \frac{\nu}{v_*}\ .\ A\ v_* = B^2\ \frac{\nu}{{v_*}^2} \tag{33}$$

The transient heating of the fluid element (which is being renewed from the main stream) by conduction for a length of time T under temperature difference θ at time $t = 0$ is given by the equation (reference 4):

$$\vartheta = \frac{\theta}{2}\left(1 + \psi\left(\frac{y}{2\sqrt{at}}\right)\right) \tag{34}$$

A solution of $\frac{\partial \vartheta}{\partial t} = a \frac{\partial^2 \vartheta}{\partial y^2}$ which meets the boundary conditions where ψ is the Gaussian function

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx$$

From equation (34) the quantity of heat which flows to the fluid particles per unit wall-surface area is

$$Q = \int_0^T \gamma\, c_p\, a \left(\frac{\partial \vartheta}{\partial y}\right)_{y=0} dt = \theta\, \gamma\, c_p \sqrt{\frac{aT}{\pi}}$$

or the average heat rate is

$$q = \frac{Q}{T} = \frac{\theta\, \gamma\, c_p\, v_*}{B\sqrt{\pi}} \sqrt{\frac{a}{\nu}}$$

so that

$$\theta = \Delta_\vartheta \frac{q}{\gamma\, c_p\, v_*} \sqrt{Pr}$$

and by comparison with equation (30)

$$f(Pr) = \sqrt{Pr}$$

These calculations are qualitative, but they lead to a likely supposition that

$$f(Pr) = Pr^m \quad \text{where} \quad \frac{1}{2} < m < 1 \tag{35}$$

The problem of determining the mean Pr_* is yet open. One could set $Pr_* = (Pr_w + Pr_0)/2$, the mean at the wall and on the inner surface of the boundary layer (inner always means toward the wall center), but the logarithmic mean seems more basic.

In the vicinity of the wall

$$\eta = v_* \int_0^y \frac{dy}{\nu} = v_* \; y\overline{(1/\nu)}$$

and if $\frac{1}{\nu} = \frac{1}{\nu_w} e^{ky}$ (which is a likely supposition), then

$$\overline{1/\nu} = \frac{\frac{1}{\nu_o} - \frac{1}{\nu_w}}{\ln \frac{\nu_w}{\nu_o}} = \frac{\nu_w - \nu_o}{\nu_w \nu_o \ln \frac{\nu_w}{\nu_o}}$$

Since $\frac{1}{P_r} = \frac{a}{\nu}$ varies with $1/\nu$ mostly, approximately there may be written

$$1/P_{r*} = \overline{\left(\frac{1}{P_r}\right)} = \frac{P_{r_w} - P_{r_o}}{P_{r_w} P_{r_o} \ln \frac{P_{r_w}}{P_{r_o}}} \tag{36}$$

for the value to be used in $f(P_{r_*})$.

THE HEAT TRANSFER EQUATION

Referring to equations (29) and (30), there results upon summation, using the theorem expressed by equation (35):

$$\frac{\gamma \; c_p \; v_*}{q} (\vartheta - \vartheta_w) = \frac{u}{v_*} + \Delta_\vartheta \; P_{r*}{}^m - \Delta_o + \frac{1}{\chi}\left(1 - \frac{1}{P_{r_o} - 1} \ln P_{r_o}\right) \tag{37}$$

The constant Δ_ϑ is yet to be determined. Experience teaches (reference 5) that for $Pr = \text{const} \simeq 1$ (gases) the $(\vartheta - \vartheta_w)$-curves and the u-curves coincide upon appropriate change of scale - which result is accomplished analytically by placing

$$\Delta_\vartheta = \Delta_o = 9.46$$

in equation (37). On the other hand, if this constant is to have universal significance, it must hold for all values of P_r; if such is the case,

$$\frac{\gamma\ c_p\ v_*}{q}\ (\vartheta - \vartheta_w) = \frac{u}{v_*} + \Delta_o({P_r}_*^{\ m} - 1) + \frac{1}{\chi}\left(1 - \frac{1}{{P_r}_o - 1}\ \ln {P_r}_o\right) \tag{38}$$

obtaining the mean over the cross section

$$\frac{q}{\gamma\ c_p\ \bar{u}(\bar{\vartheta} - \vartheta_w)} = \frac{\left(\frac{v_*}{\bar{u}}\right)^2}{1 + \frac{v_*}{\bar{u}}\left[\Delta_o({P_r}_*^{\ m} - 1) + \frac{1}{\chi}\left(1 - \frac{1}{{P_r}_o - 1}\ \ln {P_r}_o\right)\right]}$$

which is the equation for $\frac{Nu}{Re}\ P_r$. Substituting $\Delta_o = 9.46$, $\chi = 0.406$, and

$$\lambda = \frac{2R}{\rho\frac{\bar{u}^2}{2}}\ \frac{dp}{dx} = -8\left(\frac{v_*}{\bar{u}}\right)^2$$

yields the expression for heat transfer

$$C \equiv \frac{q}{\gamma\ c_p\ \bar{u}\ (\bar{\vartheta} - \vartheta_w)}$$

$$= \frac{\lambda/8}{1 + \sqrt{\frac{\lambda}{8}}\left[9.46({P_r}_*^{\ m} - 1) + 2.46\left(1 - \frac{1}{{P_r}_o - 1}\ \ln {P_r}_o\right)\right]} \tag{39}$$

where λ is given by (reference 1, p. 133):

$$\frac{1}{\sqrt{\lambda}} = 2.005 \log Re \sqrt{\lambda} - 0.806 \tag{40}$$

an equation which is in agreement with the results of Nikuradse (reference 3) on smooth pipes.

It is not known how to compute Re in equation (40) for nonisothermal flow. The influence of viscosity is limited to the laminar sublayer on the wall side of which the eddy diffusivity ϵ is, by equation (16), of the order of magnitude of ν. Since ϵ increases greatly with y as one proceeds from the wall, and since $\epsilon + \nu$ is the combined "austausch" coefficient, it is clear that the value ν_0 on the wall side of the boundary layer will fix the resistance to flow. Until better data are available, it will suffice to set

$$Re = Re_0 = \frac{2R\bar{u}}{\nu_0}$$

into equation (40), Utilizing

$$m = 0.77 \tag{41}$$

equation (39) holds well for heating and cooling.

Actually, equation (39) contains properties which are not known as the result of experiment; for instance, Pr_0 and ν_0, which are fixed by the temperature of the sublayer on the wall side ν_0 which can be fixed by knowing the rate of heat transfer. For fair values of Prandtl's modulus, or for small temperature differences $\vartheta_0 \approx \bar{\vartheta}$, since for sufficiently large magnitudes of Pr most of the temperature drop will occur in the sublayer. For small Pr (< 0.5), or for large temperature differences, one may employ equation (39) and the method of successive approximations (two will suffice). One determines an approximate q_0 setting $\vartheta_0 = \bar{\vartheta}$ (mean) in the following equation and in equation (36):

$$\theta = \vartheta_0 - \vartheta_w = 9.46 \frac{q_0}{\gamma \; c_p \; v_*} Pr_*^{\,m} \tag{42}$$

With these approximate properties, the calculation is repeated, using equations (39) and (40). Finally, the mean temperature across the section f in area is:

$$\bar{\vartheta} = \frac{1}{f} \int \vartheta \, df$$

while the mixed mean $\bar{\vartheta}_1$ is usually measured

$$\vartheta_1 = \frac{\int u \, \vartheta \, df}{\int u \, df}$$

Based on the mixed mean (reference 1, pp. 129-133):

$$\frac{\gamma\, c_p\, v_*}{q} (\bar{\vartheta}_1 - \vartheta_w) = \frac{\bar{u}}{v_*}\left(1 + 10.6 \left(\frac{v_*}{\bar{u}}\right)^a\right)$$

$$+ \Delta_0 ({P_r}_*{}^m - 1) + \frac{1}{\chi}\left(1 - \frac{1}{{P_r}_0 - 1} \ln {P_r}_0\right)$$

where $\bar{u}$ is the mean velocity across the section. Then

$$\frac{q}{\gamma\, c_p\, u(\bar{\vartheta}_1 - \vartheta_w)}$$

$$= \frac{\lambda/8}{1 + 10.6\,\frac{\lambda}{8} + \sqrt{\frac{\lambda}{8}}\left[9.46({P_r}_*{}^m - 1) + 2.46\left(1 - \frac{1}{{P_r}_0 - 1} \ln {P_r}_0\right)\right]} \tag{43}$$

APPROXIMATE EXPRESSIONS FOR PRACTICAL USE

For numerical work, the Blasius expression is useful:

$$\frac{\lambda}{8} = 0.0396\ \mathrm{Re}^{-0.25} \tag{44}$$

which holds well up to $\mathrm{Re} \approx 10^6$.

When P_r is approximately unity, one may derive

$$P_r{}^m - 1 \approx m\,(P_r - 1);\quad 1 - \frac{1}{P_r - 1} \ln P_r \approx \frac{P_r - 1}{2}$$

For $m = 0.77$ and for small temperature differences

$$\frac{\mathrm{Nu}}{\mathrm{Re}\, P_r} = \frac{0.0396\ \mathrm{Re}^{-0.25}}{1 + 1.69\ \mathrm{Re}^{-0.125}\,(P_r - 1)} \tag{45}$$

which agrees well with Prandtl except that he obtains 1.74 for the coefficient instead of 1.69. For $m = 0.79$, agreement is perfect. The problem of more accurately determining m, naturally is open.

For large magnitudes of P_r, the term $\frac{1}{P_r - 1} \ln P_r$ may be neglected. In general, one must solve the equation:

$$C \equiv \frac{Nu}{Re\ P_r}$$

$$= \frac{0.0396\ Re_0^{-0.25}}{1+0.199\ Re_0^{-0.125}\left[9.46(P_r^{\ 0.77}-1)+2.46\left(1-\frac{1}{P_{r_0}-1}\ln P_{r_0}\right)\right]} \tag{46}$$

Kraussold (reference 6) has proposed an empirical expression:

$$\frac{Nu}{Re\ Pr} = 0.024\ Re^{-0.2}\ P_r^{-n} \tag{47}$$

where $n = 0.63$ for heating, and $n = 0.7$ for cooling. It is not uninteresting to compare the two formulas. (See the following table.) The right side of equation (46) is computed for $P_{r_*} = P_{r_0}$, which will hold for small temperature differences.

Pr	Nu/(Re P_r)		
	Equation (46)	Equation (47) Heating n = 0.63	Equation (47) Cooling n = 0.7
1	0.00596	0.00380	0.00380
5	.00155	.00138	.00123
10	.000984	.000892	.000759
50	.000313	.000323	.000246
100	.000187	.000209	.000151

It is seen that for large Pr the results from equation (46) lie between those for heating and cooling.

To illustrate the method of carrying out the computations, we will now compute the unit conductance for water corresponding to a mean temperature of 50° C, a mean velocity of 100 centimeters per second through a tube of 3 centimeters diameter for a wall temperature of 100° C.

As a first approximation, we will choose the properties for $\vartheta = 50^\circ$; $\nu_0 = 0.00562$ cm²/sec., $Pr_0 = 3.58$. For $\vartheta = 100^\circ$ C, $Pr_w = 1.72$, then $Pr_* = 2.43$ and $Re_0 = 53,400$. From equation (46), $C = 0.001696$, and from equation (42), $\vartheta_w - \vartheta_0 = 31^\circ$. Therefore, $\vartheta_0 = 69^\circ$ C.

The properties corresponding to the second approximation are: $\nu_0 = 0.00419$, $Pr_0 = 2.57$; $Pr_w = 1.72$. $Pr_* = 2.09$, $Re_0 = 71,600$, and from (46)

$$C = \frac{q}{\gamma\ c_p\ \overline{u}\ (\vartheta - \vartheta_w)} = 0.001724$$

For $\gamma = 988$ kg/m³, $c_p = 0.998$ kcal/kg° C:

$$\alpha = \frac{q}{\vartheta - \vartheta_w} = 3600 \times 0.001724 \times 1 \times 988 \times 0.998 = 6120 \text{ kcal/m}^2\text{h}^\circ \text{ C}$$

ROUGH TUBES

Tubes may be thought of as completely rough if $\frac{v_* k}{\nu} > 100$ (references 1 and 7). Actually, values of over 60 will satisfy the requirement. k is the mean height of the wall asperities; still better is the equivalent height of sand roughness. In this case, the resistance equation is quadratic where

$$\frac{1}{\sqrt{\lambda}} = 2.005 \log \frac{R}{k} + 1.73 \qquad (48)$$

This equation has been established by experiment and was derived by the author on the assumption that the boundary-layer thickness (in which the wall transfer mechanism transpires) is equal to k. Applying this postulate to the thermal problem in a manner similar to that obtained for a smooth tube yields

$$\theta = \Delta_r \frac{q}{\gamma\ c_p\ v_*} \frac{v_* k}{a} \quad \text{and} \quad \Delta_r \frac{q}{\gamma\ c_p\ v_*} \sqrt{\frac{v_* k}{a}}$$

where the mean time of repose of the fluid particle in the boundary layer is now proportional to k/v_*. On the same basis as was used previously:

$$\theta = \vartheta_0 - \vartheta_w = \Delta_r \frac{q}{\gamma\, c_p\, v_*} \left(\frac{v_* k}{a}\right)^n ; \quad \frac{1}{2} < n < 1 \qquad (49)$$

For a completely rough wall, the kinematic viscosity ν and the thermal diffusivity a may be neglected throughout the whole tube, and in place of equation (29), the following is obtained:

$$\frac{\gamma\, c_p\, v_*}{q} (\vartheta - \vartheta_0) = \frac{u}{v_*} - \Delta \qquad (50)$$

where $\Delta = 7.7$ for $\beta = 3.5$ yields the best fit (reference 1, p. 277).

From equations (49) and (50), it follows that

$$\frac{\gamma\, c_p\, v_*}{q} (\vartheta - \vartheta_w) = \frac{u}{v_*} + \Delta_r \left(\frac{v_* k}{a}\right)^n - \Delta \qquad (51)$$

Averaging (not mixed mean) over the cross section, and introducing the function factor yields

$$\frac{q}{\gamma\, c_p\, \bar{u}\, (\bar{\vartheta} - \vartheta_w)} = \frac{\lambda/8}{1 + \sqrt{\frac{\lambda}{8}} \left[\Delta_r \left(\frac{k v_*}{a}\right)^n - \Delta\right]} \qquad (52)$$

There are no data as yet against which this expression can be checked. Pohl's (reference 8) measurements extend only to $\frac{kv_*}{\nu} = 12$ as the formula demands, while equation (52) is good only for $\frac{kv_*}{\nu} > 60$. But from these measurements, it is safe to state that the term in square brackets in the denomination of the right side increases with increase $\frac{kv_*}{a}$ as the formula demands, while it is independent thereof in smooth tubes. (See equation (39).)

SUMMARY

The heat transfer accompanying turbulent flow in tubes has been treated by a new theory of wall turbulence, and a formula for smooth tubes has been derived (equation (39)) which is asymptotic at $Re \longrightarrow \infty$. It agrees very well with the data available to date. The formula also holds for the flow along a flat plate if λ is based on the velocity far away. For rough tubes, the unit conductance is shown to be a function of $k\, v_* / \nu$; the two empirical constants (Δ_r, n) which appear in equation (52) cannot yet be determined because of lack of experimental data.

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